Lorentz Transform of Black Body Radiation Temperature

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Abstract. - The Lorentz transform of black body radiation has been investigated from the view point of relativistic statistical mechanics. The result shows that the well known expression with the directional temperature can be derived based on the inverse temperature four vector. The directional temperature in the past literature was the result of mathematical manipulation and its physical meaning is not clear. The inverse temperature four vector has, in contrast, clear meaning to understand relativistic thermodynamical processes.

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Introduction. – It is well known that black body radiation obeys the Planck distribution; the following expression can be found in textbooks:

$$n(\omega)\,d\omega = \frac{\omega^2}{2\pi^2[\exp(\omega/T)-1]}\,d\omega\,, \tag{1}$$

where n is the number density of photons with frequency ω and T is the temperature. (The unit system is such that light speed = Boltzmann constant = Planck constant =1.) The above formula is the expression in the reference frame of the black body cavity. The expression for an observer moving relative to the cavity has been calculated in the context of the cosmic microwave background (CMB).

If the solar system is moving relative to the CMB's rest frame, the distribution of CMB observed at the earth will be different from (1) and the difference can tell us the motion of the solar system. Several authors published the same result in the same year [1–3]; the number density of photons coming from the solid angle Ω with frequency ω is expressed as

$$n(\omega,\Omega)\,d\omega d\Omega = \frac{\omega^2}{2\pi^2[\exp(\omega/T_{\rm eff}(\theta))-1]}\,d\omega d\Omega\,. \tag{2}$$

In the above expression $T_{\rm eff}$ is the effective temperature, which is called "directional temperature", defined as

$$T_{\rm eff}(\theta) = \frac{T_*\sqrt{1-V^2}}{1-V\cos(\theta)}\,, \tag{3}$$

where T_* is the black body temperature as measured in the cavity rest frame [T in (1)], V is the observer's velocity, and θ is the angle between the observer's motion and the direction of observation.

The above result has been obtained from a purely mathematical manipulation, and no thermodynamical consideration is used to derive it. Therefore, $T_{\rm eff}$ is just a mathematical

shorthand in the formula and it is not clear whether T_{eff} has the meaning of temperature or not. This is sufficient for its original purpose, that is, to determine the motion of the solar system relative to CMB (e.g., [4]). However, it is not enough when we wish to investigate the thermodynamical properties of the black body radiation. For example, suppose a moving matter that is immersed into a black body radiation. When the matter has proper temperature (temperature measured in the comoving frame) T such that $T_{\text{eff}}(0) < T < T_{\text{eff}}(\pi/2)$, then we cannot tell the direction of heat flow; usually heat flows from the higher temperature to the lower temperature, but from knowing T_{eff} it cannot be determined which has the higher temperature.

An attempt to understand the thermodynamics of moving black body radiation has been made in relatively recent years. Aldrovandi and Gariel [5] regarded (2) as the temperature transformation law and concluded that the temperature of a moving object becomes higher. Costa and Matsas [6] calculated the photon distribution using the Unruh-DeWitt detector moving relative to the radiation, and showed the equilibrium distribution does not have the form of (1). Form this fact, Landsberg and Matsas [7, 8] claimed that the relativistic temperature transformation is impossible and the concept of temperature can be defined only in the comoving reference frame. Ares de Parga et al., [9] have examined this problem based on the theory they have proposed, and concluded that the expression with the directional temperature can be understood within their theory.

What we would like to show in the present letter is that the formula (2) can be consistently derived from the view point of relativistic statistical mechanics. The directional temperature in (3) is just a shorthand notation, and does not have thermodynamical implication as a temperature. In contrast, the calculation here derives the same expression as (3) based on the inverse temperature four vector proposed in the context of relativistic thermodynamics. This inverse temperature four vector has clear thermodynamical meaning because it comes from the conservation law of energy-momentum, just in the same way as the inverse temperature in conventional non-relativistic statistical mechanics.

The inverse temperature four vector was originally introduced by van-Kampen [10] in the controversy on relativistic thermodynamics in 1960s, and later refined by Israel in a more transparent form [11]. There are a number of different formulations of relativistic thermodynamics (see, e.g., [12]), however, it can be shown that the other formulations can be derived from the van Kampen-Israel theory with the inverse temperature four vector [13]. We will see in the present paper that the inverse temperature four vector can be also applied for the covariant treatment of black body radiation.

Momentum in Statistical Mechanics. – Before exploring the black body radiation in relativity, we briefly demonstrate our tactics with a simple example of non-relativistic classical ideal gas. Let $f(\mathbf{p})$ be the single particle distribution function as a function of momentum $\mathbf{p}$. We employ the maximum entropy approach (e.g., [14]) to calculate the equilibrium distribution, which is obtained by maximizing the following entropy

$$S = \int f \ln f \, d\mathbf{p}\,, \tag{4}$$

under the following constraints of the particle number and energy conservation,

$$\int f(\mathbf{p})\, d\mathbf{p} = 1,\ \ N \int \frac{1}{2m}\mathbf{p}^2 f(\mathbf{p})\, d\mathbf{p} \quad = \quad \text{total energy}, \tag{5}$$

where m is the mass of a particle and N is the total number of particles. The equilibrium distribution is obtained as

$$f(\mathbf{p}) \propto \exp\left(\alpha - \frac{\beta \mathbf{p}^2}{2m}\right). \tag{6}$$

The parameter α and β in the above expression are the Lagrange's coefficients arising from the constraints of (5), and should be determined appropriately to satisfy the constraints.

When we observe this distribution from a frame moving with the relative velocity $\mathbf{V}$, then the Galilei transform of the distribution function can be calculated by replacing $\mathbf{p} \to \mathbf{p} - m\mathbf{V}$ as

$$f(\mathbf{p}) \propto \exp\left(\alpha - \frac{\beta}{2m}(\mathbf{p} - m\mathbf{V})^2\right) . \tag{7}$$

This expression is obtained by mathematical transform of (6) and no thermodynamical consideration is required once (6) is given; this corresponds to the derivation of (2) [1–3].

There can be another way to derive (7) from the view point of entropy maximization. We introduce another conservation law, the momentum conservation namely, in addition to the energy constraint of (5):

$$N \int \mathbf{p} f(\mathbf{p})\, d\mathbf{p} = \text{total momentum} . \tag{8}$$

Then three other Lagrange's coefficients appear corresponding to the three components of momentum, and the distribution becomes

$$f(\mathbf{p}) \propto \exp\left(\alpha' - \frac{\beta' \mathbf{p}^2}{2m} + \beta'_x p_x + \beta'_y p_y + \beta'_z p_z\right) . \tag{9}$$

The Lagrange's coefficients α', β', $\beta'_i (i = x, y, z)$ should be determined to satisfy the constraints (5) and (8). Since this distribution has the same energy and momentum as (7), the coefficients are

$$\begin{aligned} \alpha' &= \alpha + \frac{1}{2} m \mathbf{V}^2 , \\ \beta' &= \beta , \\ \beta'_i &= \beta V_i \ (i = x, y, z) , . \end{aligned} \tag{10}$$

The above result has something more than the derivation of (7). Since the coefficients β_i are obtained in a same way as to derive the inverse temperature β, they have similar meaning in thermodynamics. If two bodies with different temperature are thermally connected, in other words, there is random energy exchange between the bodies, the energy flow is such as to reduce the difference of the inverse temperature β. We can generalize this statement to the random momentum exchange between two bodies moving relative to each other. The momentum is transferred in the direction to reduce the relative velocity because it increases the total entropy. The result is the frictional force between the two bodies.

The example in this section demonstrates the role of momentum as a thermodynamical parameter; three more inverse temperature arise corresponding to the three components of momentum. This non-relativistic example may be rather trivial because the inverse temperature of energy (β) is unchanged under the Galilei transform. However, if we wish to construct covariant relativistic thermodynamics, not only energy but also the three components of momentum should be regarded as thermodynamical quantities because they are components of a four vector. Correspondingly inverse temperatures β and β_i $(i = x, y, z)$ forms a four vector when we generalize the calculation to relativity. This four vector is the covariant expression of the inverse temperature in the relativistic thermodynamics proposed by van Kampen and Israel [10, 11].

It is possible to perform the same calculation as above for a relativistic ideal gas, however, there is a subtle point in the maximum entropy calculation, and controversy is still going on [15, 16]. Therefore, we dare not examine this subject in the present letter and move onto the black body radiation in the following.

Black Body Temperature. – What we have learned form the previous section is that the Boltzmann factor $\exp(-\beta E)$ should be replaced with

$$\exp(-\beta E) \rightarrow \exp\left(-\sum_{\mu=0}^{3} \beta_\mu P^\mu\right), \tag{11}$$

where P^μ is the energy-momentum of the system, to include the momentum as a thermodynamical quantity. As we have seen in the previous section, the inverse temperatures β_μ are obtained from statistical mechanics, and thus we know their roles in thermodynamics. However, we do not know whether they are transformed as a four vector or not at this stage; we used the notation with Σ to emphasize this point in the above expression.

In the case of a photon gas, the energy-momentum of a photon is given by its four dimensional wave number. Suppose a photon gas in a black body cavity, which is moving in the z $(= x^3)$ direction with the velocity V in one reference frame. The x and y components of the momentum vanish because of the symmetry, so we can set $\beta_x = \beta_y = 0$. Following the standard procedure in the statistical mechanics, the number of photons in one wave mode is obtained as

$$N_i = \frac{1}{\exp(\beta_t \omega_i - \beta_z k_{iz}) - 1}, \tag{12}$$

where ω_i and k_{iz} are the frequency and wave number of the i-th mode.

Let us introduce polar coordinates $(r, \theta, \phi)$to calculate waves propagating in one direction; we choose the coordinates such that $\theta = 0$ is the direction of the spatial wave vector. The number of photons in the limit of continuous frequency can be calculated in the same way as to derive (1), which yields

$$n(\omega, \Omega)\, d\omega d\Omega = \frac{\omega^2}{2\pi^2[\exp[(\beta_t - \beta_z \cos\theta)\omega] - 1]}\, d\omega d\Omega\,. \tag{13}$$

We have used the dispersion relation of photons $\omega^2 = k_z^2$ to obtain the above expression. The inverse temperature β_μ should be determined such that the distribution $n(\omega, \Omega)$ gives the total energy-momentum correctly. This can be done in the same way as we have done in deriving (10). Comparing (2) and (13) we obtain

$$\beta_t = \frac{1}{T_* \sqrt{1 - V^2}}\,, \quad \beta_z = \frac{V}{T_* \sqrt{1 - V^2}}\,. \tag{14}$$

This result can be generalized to a covariant form as

$$\beta_\mu = \frac{u_\mu}{T_*}\,, \tag{15}$$

where u_μ is the relative four velocity between the radiation and the observer. We understand the inverse temperatures β_μ form a four vector from the above explicit form. This is the inverse temperature four vector in the van Kampen-Israel theory [10, 11]. Unlike the directional temperature T_{eff}, the above four vector β_μ has been derived from the view point of statistical mechanics. Therefore β_μ has clear meaning as inverse temperatures, and can tell the direction of the thermal energy-momentum exchange as being discussed by van Kampen [10].

Concluding Remarks. – Brief remarks are to be made on the past literature before closing this letter. Aldrovandi [5] examined the temperature transformation assuming the directional temperature has the thermodynamical meaning somehow; he did not give the reason for this assumption as he states "we prefer to avoid an 'inside' thermodynamical discussion." Ares de Parga et al., [9] have their own reasoning to interpret the directional

temperature within the theory they proposed. The two results seems to contradict each other: the former suggests the higher temperature for a moving body whereas the latter predicts it lower. This contradiction is quite similar to the controversy on the relativistic thermodynamics in 1960s. Both may be consistent within each framework, however, the author of this paper believes the expression with inverse temperature four vector is clear and transparent.

Costa and Matsas [6] have calculated the particle distribution of photons using the Unruh-DeWitt detector and obtained the following expression,

$$n(\omega) = \frac{T_* \sqrt{1-v^2}}{4\pi v} \ln\left(\frac{1-\exp(-\omega\sqrt{1+v}/T_0\sqrt{1-v})}{1-\exp(-\omega\sqrt{1-v}/T_0\sqrt{1+v})}\right). \tag{16}$$

The expression can be also obtained by integrating (2) over the solid angle; this can be understood because the Unruh-DeWitt detector measures energy only and directional dependence is smeared out. The above expression does not have the form of the Planck distribution, and from this fact Landsberg and Matsas [7,8] argued thatLorentz transforming the temperature would be impossible.

However, a distribution function changes its shape in general when expressed as a function of energy only. For example, suppose a non-relativistic gas with distribution function (7). When we express the distribution as a function of energy ignoring directional dependence, we obtain

$$f(E) = \int f(\mathbf{p})\,\delta\left(E - \frac{1}{2m}\mathbf{p}^2\right) d\mathbf{p} = \frac{T}{V_0\sqrt{2\pi E}} \sinh\left(\frac{mV_0^2}{T}\right) \exp\left(-\frac{E}{T}\right), \tag{17}$$

which does not have the form of Boltzmann distribution $f \propto \exp(-\beta E)/\sqrt{E}$; obviously this does not mean the concept of temperature T is invalid in the Galilei transform.

Similarly the concept of temperature (or inverse temperature) is still valid even when (16) does not have the form of the Planck distribution. What Landsberg and Matsas [7,8] argued should be understood that the Lorentz transform of temperature is impossible when one tries to express the temperature by a single value; it is possible when we treat the inverse temperature as a four vector. The present letter has shown that we can interpret the distribution (2) as the statistical equilibrium state with the inverse temperature four vector β_μ in the van Kampen-Israel theory [10,11].

REFERENCES

[1] T. N. Bracewell, E. K. Conklin, An observer moving in the 3 k radiation field, Nature 219 (1968) 1343–1344. doi:10.1038/2191343a0.

[2] G. R. Henry, R. B. Feduniak, J. E. Silver, M. A. Peterson, Distribution of blackbody cavity radiation in a moving frame of reference, Phys. Rev. 176 (5) (1968) 1451–1455. doi:10.1103/PhysRev.176.1451.

[3] P. J. E. Peebles, D. T. Wilkinson, Comment on the anisotropy of the primeval fireball, Phys. Rev. 174 (5) (1968) 2168. doi:10.1103/PhysRev.174.2168.

[4] A. Kogut, C. Lineweaver, G. F. Smoot, C. L. Bennett, A. Banday, N. W. Boggess, E. S. Cheng, G. de Amici, D. J. Fixsen, G. Hinshaw, P. D. Jackson, M. Janssen, P. Keegstra, K. Loewenstein, P. Lubin, J. C. Mather, L. Tenorio, R. Weiss, D. T. Wilkinson, E. L. Wright, Dipole Anisotropy in the COBE Differential Microwave Radiometers First-Year Sky Maps, Astrophys J 419 (1993) 1–+. arXiv:astro-ph/9312056, doi:10.1086/173453.

[5] R. Aldrovandi, J. Gariel, On the riddle of the moving thermometers, Physics Letters A 170 (1992) 5–10. doi:10.1016/0375-9601(92)90382-V.

[6] S. S. Costa, G. E. A. Matsas, Temperature and relativity, Physics Letters A 209 (1995) 155–159. arXiv:gr-qc/9505045, doi:10.1016/0375-9601(95)00843-7.

[7] P. T. Landsberg, G. E. A. Matsas, Laying the ghost of the relativistic temperature transformation, Physics Letters A 223 (1996) 401–403. arXiv:physics/9610016, doi:10.1016/S0375-9601(96)00791-8.
[8] P. T. Landsberg, G. E. A. Matsas, The impossibility of a universal relativistic temperature transformation, Physica A Statistical Mechanics and its Applications 340 (2004) 92–94. doi:10.1016/j.physa.2004.03.081.
[9] G. Ares de Parga, B. López-Carrera, F. Angulo-Brown, A proposal for relativistic transformations in thermodynamics, Journal of Physics A Mathematical General 38 (2005) 2821–2834. doi:10.1088/0305-4470/38/13/001.
[10] N. G. van Kampen, Relativistic Thermodynamics of Moving Systems, Physical Review 173 (1968) 295–301. doi:10.1103/PhysRev.173.295.
[11] W. Israel, Nonstationary irreversible thermodynamics: A causal relativistic theory, Annals of Physics 100 (1976) 310–331. doi:10.1016/0003-4916(76)90064-6.
[12] C. K. Yuen, Lorentz Transformation of Thermodynamic Quantities, American Journal of Physics 38 (1970) 246–252. doi:10.1119/1.1976295.
[13] T. K. Nakamura, Three Views of a Secret in Relativistic ThermodynamicsarXiv:0812.3725.
[14] R. D. Rozenkrantz (Ed.), Papers on probability statistics and statistical physics, Kluwer: Dordrecht, 1983.
[15] J. Dunkel, P. Talkner, P. Hänggi, Relative entropy, Haar measures and relativistic canonical velocity distributions, New Journal of Physics 9 (2007) 144–+. arXiv:cond-mat/0610045, doi:10.1088/1367-2630/9/5/144.
[16] T. K. Nakamura, Relativistic Equilibrium Distribution by Relative Entropy Maximization-arXiv:0909.2732.